ON APPLICATIONS OF THE MODEL SPACES TO THE CONSTRUCTION OF COCYCLIC PERTURBATIONS OF THE SEMIGROUP OF SHIFTS ON THE SEMIAXIS

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Abstract. We describe a construction of cocyclic perturbations of the semigroup of shifts on the semiaxis by means of the theory of model spaces. It is shown that one can choose an inner function that determines the model space so that the elements of the perturbed semigroup have a prescribed spectral type and differ from the elements of the initial semigroup by operators from the Schatten–von Neumann class $\mathfrak{S}_p$, $p > 1$. The case of the trace class $\mathfrak{S}_1$ perturbations is considered separately.

Keywords: semigroup of shifts, inner function, Schatten–von Neumann classes.

1. Introduction

Let us assume that $(S_t,\ t \geq 0)$, and $(\tilde{S}_t,\ t \in \mathbb{R})$ are the semigroup of shifts in the space $H = L^2(\mathbb{R}_+)$, and the group of shifts (its unitary dilation) in the space $\tilde{H} = L^2(\mathbb{R})$, defined by formulae

$$(S_t f)(x) = \begin{cases} f(x-t), & x > t, \\ 0, & 0 \leq x \leq t, \end{cases} \qquad f \in H,$$

and

$$(\tilde{S}_t g)(x) = g(x-t), \qquad g \in \tilde{H},$$

respectively. Sometimes it is convenient to assume that the multiplicative group of the algebra $B(H)$ of bounded operators in the space H is embedded in the multiplicative group of the algebra $B(\tilde{H})$ in such a way that elements $B(H)$ act on functions $f \in \tilde{H}$ with a support on the negative semiaxis as the identical mapping. In this case operators, acting in the space H, will be considered as operators in

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$\tilde{H}$ as well. A strongly continuous family of unitary operators $(W_t,\ t \geq 0)$ in the space H is called a *cocycle* of the semigroup of shifts $(S_t,\ t \geq 0)$ if the condition

$$W_{t+s} = W_t \tilde{S}_t W_s \tilde{S}_{-t}, \quad t, s \geq 0, \qquad W_0 = I \tag{1}$$

holds true (see [1]). It follows from the condition (1) that the family of isometric operators $(V_t = W_t S_t,\ t \geq 0)$ in the space H forms a semigroup (i.e. $V_{t+s} = V_t V_s$, $t, s \geq 0$), which will be called a *cocyclic perturbation* of the semigroup of shifts $(S_t,\ t \geq 0)$.

In the present paper we will show that any cocyclic perturbation of the semigroup (S_t) is unitarily equivalent to the orthogonal sum

$$(V_t) \cong (U_t \oplus S_t), \tag{2}$$

where $(U_t,\ t \geq 0)$ is a semigroup of unitary operators, and the following two theorems hold true. Here and in what follows all semigroups under consideration are supposed to be strongly continuous; the symbol $\mathfrak{S}_p$ denotes the Schatten-von Neumann operator ideals.

Theorem 1. *For any semigroup of unitary operators $(U_t,\ t \geq 0)$ possessing a spectral measure which is singular with respect to the Lebesgue measure, there is a cocycle $(W_t,\ t \geq 0)$, satisfying the condition*

$$W_t - I \in \mathfrak{S}_p$$

for all $p > 1$, where (2) holds true for the cocyclic perturbations $(V_t = W_t S_t,\ t \geq 0)$, and

$$V_t - S_t \in \mathfrak{S}_1, \qquad t \geq 0. \tag{3}$$

As a corollary of Theorem 1 we obtain analogous results for an arbitrary (not necessarily singular) spectral measure.

Theorem 2. *For any semigroup of unitary operators $(U_t,\ t \geq 0)$ and for any $p > 1$ there is a cocycle $(W_t,\ t \geq 0)$, satisfying the condition*

$$W_t - I \in \mathfrak{S}_p$$

for all $p > 1$, when the relation (2) holds true for the cocyclic perturbation $(V_t = W_t S_t,\ t \geq 0)$.

In what follows it will be shown (Proposition 10) that the condition $W_t - I \in \mathfrak{S}_1$ never holds true in the considered model of cocyclic perturbations. Thus, the results of the work are in a sense optimal. It is natural to suppose, that this fact holds true in the general case.

Conjecture. For any cocycle $(W_t,\ t \geq 0)$, such that $W_t - I \in \mathfrak{S}_1$ for all $t \geq 0$, the perturbed semigroup $(V_t = W_t S_t,\ t \geq 0)$ is unitarily equivalent to the initial one: $(V_t) \cong (S_t)$.

We should note that the problem of the Markov cocyclic perturbations of the group of unitary operators connected with the questions under consideration is posed in [2], and the Markov cocycles possessing the property $W_t - I \in \mathfrak{S}_2,\ t \geq 0$ are considered in [3, 4]. The property (3) was considered in the article [5], where perturbations $(V_t,\ t \geq 0)$ of semigroup shifts $(S_t,\ t \geq 0)$ such that $V_t - S_t \in \mathfrak{S}_p,\ p \geq 1$, were investigated. A distinguishing feature of the present paper is that the considered perturbations possess additional cocyclic properties demanding consideration of unitary dilations of semigroups. The technique applied here is analogous to the one in paper [5].

2. Cocyclic perturbations of the general form

For any strongly continuous semigroup of isometric operators $(V_t,\ t \geq 0)$ in the Hilbert space H, the Wold–Kolmogorov decomposition is defined as follows:

$$H = H_0 \oplus H_1,$$

$$V_t = U_t \oplus R_t, \qquad t \geq 0, \tag{4}$$

where $(U_t,\ t \geq 0)$ is a semigroup of unitary operators in H_0, and $(R_t,\ t \geq 0)$ is a semigroup of completely nonunitary isometric operators in H_1, i.e. lacking nontrivial invariant subspaces, where they act as unitary operators.

Proposition 3. *Let the semigroup of isometric operators* $(V_t,\ t \geq 0)$ *be a cocyclic perturbation of the semigroup of shifts* $(S_t,\ t \geq 0)$*. Then the completely nonunitary part* $(R_t,\ t \geq 0)$ *in the Wold–Kolmogorov decomposition* (4) *is unitarily equivalent to the semigroup of shifts* $(S_t,\ t \geq 0)$*.*

Remark. This statement holds true for an arbitrary semigroup (not necessarily being a cocyclic perturbation) of isometric operators $(V_t,\ t \geq 0)$, if we require that $V_t - S_t \in \mathfrak{S}_p$, $p \geq 1$ (see [5]).

Proof. Let us define elements $\xi_t \in H,\ t \geq 0$, by the formula

$$\xi_t(x) = \begin{cases} 1, & 0 \leq x \leq t, \\ 0, & x > t. \end{cases}$$

Note, that the family $(\xi_t,\ t \geq 0)$ satisfies the so-called condition of an *additive cocycle* of the semigroup $(S_t,\ t \geq 0)$, i.e.

$$\xi_{t+s} = \xi_t + S_t \xi_s, \qquad s, t \geq 0,$$

and the functions $\xi_{t_1} - \xi_{s_1}$ and $\xi_{t_2} - \xi_{s_2}$ are orthogonal, if $(s_1, t_1) \cap (s_2, t_2) = \emptyset$. Moreover, linear combinations of elements $(\xi_s,\ 0 \leq s \leq t)$ generate $\operatorname{Ker} S_t^*$. Assume that $\tilde{\xi}_t = W_t \xi_t,\ t \geq 0$. To prove Proposition 3 it is sufficient to verify that for the cocyclic perturbation $(V_t = W_t S_t,\ t \geq 0)$ the family of elements $\tilde{\xi}_t$ has the following properties:

(i) $\tilde{\xi}_{t+s} = \tilde{\xi}_t + V_t \tilde{\xi}_s,\ s, t \geq 0,$

(ii) $\tilde{\xi}_{t_1} - \tilde{\xi}_{s_1}$ and $\tilde{\xi}_{t_2} - \tilde{\xi}_{s_2}$ are orthogonal, if $(s_1, t_1) \cap (s_2, t_2) = \emptyset$,

(iii) linear combinations $(\tilde{\xi}_s,\ 0 \le s \le t)$ generate $\mathrm{Ker}\, V_t^*$.

Indeed, in this case the restriction of the semigroup $(V_t,\ t \ge 0)$ to the subspace H_0, generated by $\mathrm{Ker} V_t^*,\ t \ge 0$, is unitarily equivalent $(S_t,\ t \ge 0)$, but the restriction $V_t|_{H_0^\perp}$ will be a unitary operator, because $\mathrm{Ker}\, V_t|_{H_0^\perp} = \{0\},\ t \ge 0$.

We have

$$\tilde{\xi}_{t+s} = W_{t+s}\xi_{t+s} = W_t \tilde{S}_t W_s \tilde{S}_{-t} \xi_t + W_t \tilde{S}_t (W_s) \tilde{S}_{-t} S_t \xi_s. \tag{5}$$

Note, that

$$\tilde{S}_t W_s \tilde{S}_{-t} \xi_s = \xi_s, \tag{6}$$

whereas $W_s f = f$ for the function with the support $\mathrm{supp}\, f \subset \mathbb{R}_-$. On the other hand,

$$\tilde{S}_t W_s \tilde{S}_{-t} S_t \xi_s = \tilde{S}_t W_s \xi_s = S_t \tilde{\xi}_s. \tag{7}$$

Substituting the relations (6) and (7) into the equality (5), we obtain the property (i).

Next,

$$W_{t+s}\xi_t = W_t \tilde{S}_t (W_s) \tilde{S}_{-t} \xi_t = W_t \xi_t, \qquad s, t \ge 0, \tag{8}$$

according to (6). Let $\tilde{t} = \max(t_1, t_2)$; then, taking into account (8), we obtain

$$\begin{aligned}(\tilde{\xi}_{t_1} - \tilde{\xi}_{s_1}, \tilde{\xi}_{t_2} - \tilde{\xi}_{s_2}) &= (W_{t_1}\xi_{t_1} - W_{s_1}\xi_{s_1}, W_{t_2}\xi_{t_2} - W_{s_2}\xi_{s_2}) \\ &= (W_{\tilde{t}}\xi_{t_1} - W_{\tilde{t}}\xi_{s_1}, W_{\tilde{t}}\xi_{t_2} - W_{\tilde{t}}\xi_{s_2}) = (\xi_{t_1} - \xi_{s_1}, \xi_{t_2} - \xi_{s_2}) = 0,\end{aligned}$$

if $(s_1, t_1) \cap (s_2, t_2) = \emptyset$. Thus, the property (ii) is established as well.

Finally, let us consider the equation

$$V_t^* f = S_t^* W_t^* f = 0. \tag{9}$$

It follows from (9) that $\mathrm{supp}\, W_t^* f \subset [0, t]$. Hence, f belongs to the closure of the linear envelope of elements $(W_t \xi_s,\ 0 \le s \le t)$. Since $s \le t$, we have $W_t \xi_s = W_s \xi_s = \tilde{\xi}_s$ for such elements by virtue of the relations (8). This completes the proof of the property (iii) and of the proposition. □

The next property is necessary for the construction of a model of cocycles.

Proposition 4. *Let $(V_t,\ t \ge 0)$ be the cocyclic perturbation of the semigroup of shifts $(S_t,\ t \ge 0)$ by the cocycle $(W_t,\ t \ge 0)$. Then, defining the family of the unitary operators $(W_{-t},\ t \ge 0)$ in the space $\tilde{H}$ by the formula*

$$W_{-t} = \tilde{S}_{-t} W_t^* S_t, \qquad t \ge 0, \tag{10}$$

we obtain that the family of operators $(\tilde{V}_t,\ t \in \mathbb{R})$, where

$$\tilde{V}_t = W_t \tilde{S}_t,$$

generates a group of unitary operators in the space $\tilde{H}$, and

$$\tilde{V}_t f = \begin{cases} V_t f, & \operatorname{supp} f \subset \mathbb{R}_+, \quad t \geq 0, \\ \tilde{S}_t f, & \operatorname{supp} f \subset \mathbb{R}_-, \quad t \leq 0. \end{cases}$$

Proof. As usual, let us assume that actions of the unitary operators $W_t,\ t \geq 0$, fixed initially in the space H, are extended by the identical action on f with $\operatorname{supp} f \subset \mathbb{R}_-$. Then the formula (10) provides the prolongation of the family $(W_t,\ t \geq 0)$ of unitary operators in $\tilde{H}$ for negative values of the parameter t. Moreover, the property of the cocycle

$$W_{t+s} = W_t \tilde{S}_t W_s \tilde{S}_{-t}, \qquad s, t \in \mathbb{R}$$

holds, which follows from the formula

$$I = W_{-t+t} = W_{-t} \tilde{S}_{-t} W_t \tilde{S}_t, \qquad t \geq 0,$$

resulting from the definition (10). To complete the proof it should be noted, that for $\operatorname{supp} f \subset \mathbb{R}_-$,

$$\tilde{V}_{-t} f = W_{-t} \tilde{S}_{-t} f = \tilde{S}_{-t} W_t^* f = \tilde{S}_{-t} f, \qquad t \geq 0.$$

□

3. Model of the cocyclic perturbation based on the cogenerator of the semigroup

We will need some well-known background from the theory of one-parameter semigroups (see [6]). A symmetric (probably unbounded) operator $A = s-\lim_{t\to 0+} \frac{V_t - I}{it}$ is called the generator of a strongly continuous group of the isometric operators $(V_t,\ t \geq 0)$. An isometric operator $V = (A - iI)(A + iI)^{-1}$ is called a cogenerator of a semigroup. For an isometric operator to be a cogenerator of some isometric semigroup it is necessary and sufficient that the number 1 should not belong to its point spectrum. The initial semigroup will consist of unitary operators only if A is a self-adjoint operator, or, equivalently, when V is a unitary operator such that the point 1 does not belong to its point spectrum. If we introduce the functions

$$\varphi_t(z) = \exp\left(t\frac{z+1}{z-1}\right), \qquad t \geq 0, \tag{11}$$

then the semigroup is recovered by means of the cogenerator V as follows: $V_t = \varphi_t(V),\ t \geq 0$. Let us note, that the functions φ_t are bounded and analytic in the unit disc $\mathbb{D}$.

One can readily show that the cogenerator of the semigroup of shift operators $(S_t,\ t \geq 0)$ in the space H is unitarily equivalent to the operator of the (unilateral) shift S in the Hardy space $K = H^2(\mathbb{D})$, consisting of analytical in the circle $\mathbb{D}$ functions $f(z) = \sum_{n=0}^{+\infty} c_n z^n$, for which $\sum_{n=0}^{+\infty} |c_n|^2 = \|f\|^2_{L^2(\mathbb{T})} < +\infty$. Therefore,

the Hardy space in the disc is naturally embedded in the space $\tilde{K} = L^2(\mathbb{T})$ on the circle $\mathbb{T}$. The operator of the shift S in the Hardy space is given by the formula

$$(Sf)(z) = zf(z), \qquad f \in K. \tag{12}$$

Analogously, the cogenerator of the group of shifts in the space $\tilde{H}$ is unitarily equivalent to the operator of the (bilateral) shift $(\tilde{S}f)(z) = zf(z)$ in the space $\tilde{K}$, with the operator $\tilde{S}$, apparently, being a unitary dilation of the operator S.

Assume, that E is a nontrivial invariant subspace of the shift operator S, that is, $SE \subset E$. Then, according to the Beurling theorem (see [7]), $E = \theta H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$ (i.e., the function which is analytic and bounded in the unit disc $\mathbb{D}$ with nontangential boundary values such that $|\theta(z)| = 1$ almost everywhere on $\mathbb{T}$). The orthogonal complement $K_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D}) = E^\perp$ is usually called a *model* space. The next proposition describes the model of cocyclic perturbation, applied in the present paper.

Proposition 5. *A cogenerator of any cocyclic perturbation of the semigroup of shifts on the semiaxis is unitarily equivalent to the isometric operator V in the space $K = H^2(\mathbb{D})$, for which there is an inner function θ, such that*

$$V = U \oplus S|_E, \tag{13}$$

where $S|_E$ is the restriction of the operator of the shift S to the invariant space defined by the function θ, and U is a unitary operator in the model space K_θ, which is the cogenerator of the unitary part of the Wold–Kolmogorov decomposition of the cocyclic perturbation.

Proof. For the cogenerator of the cocyclic perturbation V in the space K, one has a determined Wold–Kolmogorov decomposition $K = K_0 \oplus K_1$ such that $V|_{K_0}$ is a unitary operator and the restriction $V|_{K_1}$ is a completely nonunitary isometric operator. It follows from Proposition 3 that the restriction $V|_{K_1}$ is unitarily equivalent to the shift operator S. Therefore, in our model situation, one can use as $V|_{K_1}$ the restriction $S|_E$ to any invariant subspace E, selected so that the equality $\dim K_\theta = \dim K_0$ holds true for the corresponding model space $K_\theta = E^\perp$, which completes the proof. □

The following statement results directly from Proposition 5.

Corollary 6. *The cogenerator of the semigroup of unitary operators $(\tilde{V}_t,\ t \geq 0)$, defining the cocycle according to Proposition 4, is unitary equivalent to the operator $\tilde{V}$ in the space $\tilde{K} = L^2(\mathbb{T})$, possessing the properties*

$$\begin{aligned} \tilde{V}f &= Vf, && f \in K = H^2(\mathbb{D}), \\ (\tilde{V}^* f)(z) &= \overline{z} f(z), && f \in \tilde{K} \ominus K = L^2(\mathbb{T}) \ominus H^2(\mathbb{D}). \end{aligned}$$

4. Perturbation model based on the Clark measures

Let U be the unitary part in the Wold–Kolmogorov decomposition (13) of the cogenerator of the cocyclic perturbation. In this section we will be interested in the case when U is unitarily equivalent to the operator of multiplication by z in the space $L^2(\mu)$, with the measure μ being singular with respect to the Lebesgue measure. Note, that U a cogenerator of the semigroup according to the condition and therefore the number 1 does not belong to its point spectrum. Operators of multiplication by z in the spaces $L^2(\mu)$ and $L^2(\tilde{\mu})$ are unitarily equivalent, if the measures $\tilde{\mu}$ and μ are mutually absolutely continuous. Multiplying the measure μ by a positive weight, one can make it satisfy the following auxiliary condition, taking an important part in what follows:

$$\int\limits_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^q} < +\infty \tag{14}$$

for some $q > 3$.

Let μ be a finite singular Borel measure on the unit circle. Define the inner function θ by the formula

$$\frac{1+\theta(z)}{1-\theta(z)} = \int\limits_{\mathbb{T}} \frac{\xi+z}{\xi-z} d\mu(\xi). \tag{15}$$

Then the operator Ω, given on $L^2(\mu)$ by the formula

$$(\Omega f)(z) = (1-\theta(z)) \int\limits_{\mathbb{T}} \frac{f(\xi)d\mu(\xi)}{1-\overline{\xi}z}, \tag{16}$$

is a unitary operator from $L^2(\mu)$ onto K_θ. Moreover, the unitary operator U in $L^2(\mu)$ transforms into the unitary operator $\tilde{U}$ in the model space K_θ such that

$$\tilde{U}f = \Omega U \Omega^* f = zf + (f,g)(1-\theta),\ f \in K_\theta, \tag{17}$$

where

$$g(z) = \frac{\theta(z)-\theta(0)}{z(1-\theta(0))} \in K_\theta,$$

and therefore the operators U and $\tilde{U}$ are unitarily equivalent, see [8].

The operator (17) is the restriction to the model space K_θ of the isometric operator V, acting in the space K by formula

$$(Vf)(z) = zf(z) + (f,g)(1-\theta(z)), \qquad f \in K. \tag{18}$$

The unitary dilation of the operator (18) will be the operator

$$(\tilde{V}f)(z) = zf(z) + (f,g)(1-\theta(z)) - (f,\overline{z})(1-\overline{\theta(1)}\theta(z)), \quad f \in \tilde{K}. \tag{19}$$

Note that

$$(\tilde{V}^* f)(z) = \overline{z}f(z), \qquad f \in \tilde{K} \ominus K.$$

Therefore, according to Proposition 5 and Corollary 6, the following statement is proved.

Proposition 7. *The formulae* (18), (19) *define the model of the cogenerator of a cocyclic perturbation in the case, when the unitary part of the cogenerator in the Wold–Kolmogorov decomposition is unitarily equivalent to the operator of multiplication by z in the space $L^2(\mu)$ with the measure μ, which is singular with respect to the Lebesgue measure.*

5. Closeness of cocyclic perturbation

Let us apply the function (11) to the model cogenerator V of the semigroup of isometric operators $(V_t,\ t \geq 0)$. The isometric operator V is the restriction of the unitary operator $\tilde{V}$ defined by the formula (19) to the space $K = H^2$. Recall that the symbols S and $\tilde{S}$ define the shift operators on K and $\tilde{K}$, respectively. Then the cocycle $(W_t,\ t \geq 0)$ satisfies the equality

$$\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) = (W_t - I)\tilde{S}_t, \quad t \geq 0.$$

Therefore, inclusion of the difference $W_t - I$ into the ideals $\mathfrak{S}_p$ proves to be equivalent to the corresponding inclusion for the differences $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})$. The properties of the operators $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})$ are determined in their turn by properties of the spectral measure μ of the unitary operator (17), i.e., by its smallness (smoothness) at the point 1.

We will need the following statement, proved in [5] (Proposition 7.2).

Proposition 8. *Let the spectral measure of the unitary operator* (17) *satisfy the condition*

$$\mathfrak{M}_q(\mu) = \int\limits_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^q} < +\infty \tag{20}$$

for some $q > 3$. Then

$$\varphi_t(V) - \varphi_t(S) \in \mathfrak{S}_1, \quad t \geq 0,$$

with

$$\|\varphi_t(V) - \varphi_t(S)\|_{\mathfrak{S}_1} \leq C_q t^{1/2} (\mathfrak{M}_q(\mu))^{1/2},$$

where the constant C_q depends only on q.

The key role in the proof of the Theorems 1 and 2 is played by the following proposition, allowing one to estimate components of the unitary dilation. In this case we are not able to obtain the inclusion of $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) \in \mathfrak{S}_1$, but the difference may belong to the ideals $\mathfrak{S}_p$ for all $p > 1$.

Proposition 9. *Let the spectral measure of the unitary operator* (17) *satisfy the condition* (20) *for some* $q > 3$. *Then*

$$\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) \in \mathfrak{S}_p, \qquad p > q' = \frac{q}{q-1}, \quad t \geq 0,$$

with

$$\|\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})\|_{\mathfrak{S}_\mathfrak{p}} \leq \omega(\mathfrak{M}_q(\mu)),$$

where ω *is a positive function such that* $\omega(r) \to 0$ *when* $r \searrow 0$.

Proof. The proof of Proposition 9 consists of several steps. At the first step we will consider components of the operator $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})$ with respect to some canonical representation of the space $\tilde{K}$ and will see that all the components, except one, belong to the ideal $\mathfrak{S}_1$ due to Proposition 8. Then, we will show that the remaining component is unitarily equivalent (after conformal transformation to the upper half-plane) to the operator of multiplication by a certain function in the Paley-Wiener space. This will allow us to reduce the problem to the question of describing measures (weights), such that the embedding operator of the Paley-Wiener space belongs to the ideal $\mathfrak{S}_p$. To complete the proof we apply a theorem due to O.G. Parfenov [9].

Step 1. Analysis of components of the unitary dilation. Let us consider the matrix of the operator $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})$ with respect to the expansion $\tilde{K} = H^2_- \oplus K_\theta \oplus \theta H^2$, where $H^2_- = L^2(\mathbb{T}) \ominus H^2$. One can readily see that all the components, except one, belong to the class $\mathfrak{S}_1$. Indeed, the statement follows from Proposition 8 for the block $K_\theta \oplus \theta H^2 \to K_\theta \oplus \theta H^2$. Proceeding to the conjugate operator, we come to the conclusion that the block $H^2_- \oplus K_\theta \to H^2_- \oplus K_\theta$ is also included into $\mathfrak{S}_1$. By its construction the component $H^2 \to H^2_-$ is equal to zero. Therefore, we only need to consider the component, corresponding to the operator $H^2_- \to \theta H^2$. Moreover, note that both operators $\varphi_t(\tilde{V})$ and $\varphi_t(\tilde{S})$ on the space $\bar{\varphi}_t H^2_-$ act as operators of multiplication by φ_t, and, consequently, $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) = 0$ on $\bar{\varphi}_t H^2_-$. It remains only to study the action of the operator $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})$ on the subspace $\bar{\varphi}_t H^2 \ominus H^2 = \bar{\varphi}_t K_{\varphi_t}$. Let us denote the restriction of the operator $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})$ to the subspace $\bar{\varphi}_t K_{\varphi_t}$ by $Q : \bar{\varphi}_t K_{\varphi_t} \to H^2$.

Step 2. Inclusion of the component Q into the ideals $\mathfrak{S}_p$. Let us show that for $v \in K_{\varphi_t}$ the following equality holds true

$$Q(\bar{\varphi}_t v) = -(1 - \overline{\theta(1)}\theta)v. \tag{21}$$

If $u \in H^2_-$, then for the arbitrary function $\varphi \in H^\infty$ there is the equality

$$P_+\varphi(\tilde{V})u = \overline{\theta(1)}\theta \cdot P_+(\varphi u), \quad P_-\varphi(\tilde{V})u = P_-(\varphi u), \tag{22}$$

where the symbols P_+ and P_- denote projectors in the space $L^2(\mathbb{T})$ on the subspace H^2 and H^2_-, respectively. Indeed, this equality is easily verified for the case when $\varphi(z) = z^n$, $n > 0$, and $u(z) = z^m$, $m < 0$. Due to its linearity and continuity

the equality (22) holds true for all $u \in H^2_-$ and $\varphi(z) = z^n$, $n > 0$. Finally, due to its linearity and $*$-weak continuity, the equality (22) holds for the arbitrary function $\varphi \in H^\infty$ as well.

Since $\varphi_t(\tilde{S})u = \varphi_t u$, the equality (22) entails

$$\big(\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})\big)u = (\overline{\theta(1)}\theta - 1) \cdot P_+(\varphi_t u), \qquad u \in H^2_-.$$

Substituting $u = \bar{\varphi}_t v$, we obtain the equality (21).

Therefore, the inclusion of $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) \in \mathfrak{S}_p$ is equivalent to the inclusion

$$M_{1-\overline{\theta}(1)\theta}|_{H^2 \ominus \varphi_t H^2} \in \mathfrak{S}_p, \tag{23}$$

where the symbol M_g denotes the operator of multiplication by the function $g \in L^\infty(\mathbb{T})$.

Step 3. Transformation into the half-plane. It will be convenient to prove the inclusion (23), making a "unitary transformation" from the unit disc into the upper half-plane $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$. Let us assume that

$$\Theta(z) = \theta\left(\frac{z-i}{z+i}\right).$$

Then $\Theta(z)$ becomes an inner function in the upper half-plane: $\Theta \in H^\infty(\mathbb{C}_+)$, and $|\Theta(x)| = 1$ for almost every $x \in \mathbb{R}$, where the values of the function Θ on the real line are considered as nontangential boundary values. Defining the measure ν on the real line by the condition

$$d\mu(\xi) = \frac{d\nu(x)}{\pi(1+x^2)}, \qquad \xi = \frac{x-i}{x+i},$$

we obtain

$$\frac{1-\Theta(z)}{1+\Theta(z)} = \frac{2}{\pi i}\int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{x}{x^2+1}\right) d\nu(x).$$

The condition (20) entails that

$$\nu(\mathbb{R}) < +\infty,$$

and there is a limit $\lim\limits_{y\to+\infty} \Theta(iy)$; let us denote it by $\Theta(\infty)$. We have $|\Theta(\infty)| = 1$ and $1 - \overline{\Theta(\infty)}\Theta \in L^2(\mathbb{R})$, with

$$\|1 - \overline{\Theta(\infty)}\Theta\|_{L^2(\mathbb{R})} = |1 - \Theta(\infty)| \cdot \sqrt{\nu(\mathbb{R})}.$$

The condition (20) is equivalent to

$$\int_{\mathbb{R}} (1+|t|)^{q-2} d\nu(t) < \infty.$$

The formula

$$(Lf)(x) = \frac{1}{\sqrt{\pi}(x+i)} f\left(\frac{x-i}{x+i}\right)$$

carries out the unitary mapping of the space $L^2(\mathbb{T})$ to $L^2(\mathbb{R})$ such that the Hardy space $H^2(\mathbb{D})$ transforms to the Hardy space $H^2(\mathbb{C}_+)$. Such a transformation turns the inclusion (23) into the relation

$$M_{1-\overline{\Theta(\infty)}\Theta}|_{\mathcal{K}} \in \mathfrak{S}_p, \tag{24}$$

where $\mathcal{K} = H^2(\mathbb{C}_+) \ominus e^{itz} H^2(\mathbb{C}_+)$. The Paley–Wiener space $\mathcal{P}W_a$ consists of all entire functions of the exponential type at most a, the restriction of which to the real line belongs to $L^2(\mathbb{R})$; and, according to the classical Paley–Wiener theorem, $\mathcal{P}W_a = e^{-iaz} H^2(\mathbb{C}_+) \ominus e^{iaz} H^2(\mathbb{C}_+)$. In this case the inclusion (24) is equivalent to the question, whether the transformation of the Paley–Wiener space $\mathcal{P}W_{t/2}$ into the space $L^2(\mathbb{R}, w(t)dt)$ on the real line with the weight $w(t) = |1 - \overline{\Theta(\infty)}\Theta(t)|^2$ belongs to $\mathfrak{S}_p$. This problem was solved in the paper [9], with the following result obtained:

Theorem (O.G. Parfenov). *For any $p > 0$ the embedding operator $\mathcal{J}$ of the space $\mathcal{P}W_a$, $a > 0$ into the space $L^2(\mathbb{R}, w(t)dt)$ belongs to the class $\mathfrak{S}_p$ if and only if*

$$\mathfrak{N}_p(w) = \sum_k \left(\int_k^{k+1} w(x)dx \right)^{p/2} < \infty. \tag{25}$$

The following estimate follows immediately from the proof of the Parfenov theorem (see also [10], where a similar result is obtained for the general model spaces):

$$\|\mathcal{J}\|^p_{\mathfrak{S}_p} \le \mathfrak{N}_p(w).$$

Step 4. Application of the Parfenov theorem. It follows from the embedding $(1-\xi)^{-q} \in L^1(\mu)$ that the functional Φ,

$$\Phi(g) = \int_{\mathbb{T}} \left(\frac{1 - \overline{\theta(1)}\theta(\xi)}{1-\xi} \right)^q g(\xi)d\xi, \qquad g \in K_\theta,$$

is bounded on K_θ, and $|\Phi(g)| \le C(q)\mathfrak{M}_q(\mu)\|g\|_2$. Note that for $q \in \mathbb{N}$ the value $\Phi(g)$ coincides with the radial limit $g^{(q-1)}(1)$ of the derivative of the order $q-1$ of the function g at the point $z = 1$.

Thus, the bounded functional Φ on K_θ is generated by the function $\left(\frac{1-\overline{\theta(1)}\theta(\xi)}{1-\xi}\right)^q \in H^2(\mathbb{D})$. Strictly speaking, the function $\left(\frac{1-\overline{\theta(1)}\theta(\xi)}{1-\xi}\right)^q$ does not belong to the space K_θ, but one can easily show that the norm of its projection to the subspace θH^2 is estimated via the norm of its projection to K_θ. Therefore,

$$\int_{\mathbb{T}} \left| \frac{1 - \overline{\theta(1)}\theta(\xi)}{1-\xi} \right|^{2q} dm(\xi) \le \omega(\mathfrak{M}_q(\mu)), \tag{26}$$

where $\omega(r) \to 0$ when $r \searrow 0$ (in fact, $\omega(r) \le C(q)r$, but the explicit form of the function ω is not important for us). Substituting the variable, we obtain

$$\int\limits_{\mathbb{R}} |1-\overline{\Theta(\infty)}\Theta(t)|^{2q}(|t|+1)^{2q-2}dt < \infty.$$

Apply the Hölder inequality, we obtain

$$\begin{aligned}\int\limits_k^{k+1} &|1-\overline{\Theta(\infty)}\Theta(t)|^2 dt \\ &\le \Bigg(\int\limits_k^{k+1} |1-\overline{\Theta(\infty)}\Theta(t)|^{2q}(|t|+1)^{2q-2}dt\Bigg)^{1/q}\Bigg(\int\limits_k^{k+1}\frac{dt}{(|t|+1)^2}\Bigg)^{1/q'} \\ &\le \frac{C^{1/q}}{(|k|+1)^{2/q'}}.\end{aligned}$$

Let us assume that $p > q'$; then

$$\sum_{k\in\mathbb{Z}}\Bigg(\int\limits_k^{k+1} |1-\overline{\Theta(\infty)}\Theta(t)|^2 dt\Bigg)^{p/2} \le C^{p/2q}\sum_{k\in\mathbb{Z}}\frac{1}{(|k|+1)^{p/q'}} < \infty.$$

Thus, invoking the estimate (26) when $p > q'$, we obtain

$$\sum_{k\in\mathbb{Z}}\Bigg(\int\limits_k^{k+1} |1-\overline{\Theta(\infty)}\Theta(t)|^2 dt\Bigg)^{p/2} \le \omega(\mathfrak{M}_q(\mu)),$$

with some function ω, $\omega(r) \searrow 0$ when $r \searrow 0$. Then, applying the Parfenov theorem, we obtain the inclusion (24). Proposition 9 is proved completely. $\square$

In the model of cocyclic perturbation considered here, the relation $W_t - I \in \mathfrak{S}_p$ is equivalent to the inclusion $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) \in \mathfrak{S}_p$. In conclusion to the section note that the difference $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S})$ cannot belong to the trace class $\mathfrak{S}_1$ for all $t \ge 0$ simultaneously.

Proposition 10. *For the class of cocyclic perturbations described in Proposition 5, the inclusion $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) \in \mathfrak{S}_1$ for all $t \ge 0$ implies that θ is a unimodular constant.*

Proof. It follows from Proposition 9, that the inclusion $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) \in \mathfrak{S}_1$ is equivalent to $\mathfrak{N}_1(|1-\overline{\Theta(\infty)}\Theta|^2) < \infty$ (see (25)). It would result in

$$\int\limits_{\mathbb{R}} |1-\overline{\Theta(\infty)}\Theta(t)|\, dt \le \sum_{k\in\mathbb{Z}}\left(\int_k^{k+1} |1-\overline{\Theta(\infty)}\Theta(t)|^2\, dt\right)^{1/2} < \infty,$$

and therefore the function $1-\overline{\Theta(\infty)}\Theta$ should belong to the Hardy space H^1. But then $\int_{\mathbb{R}}\big(1-\overline{\Theta(\infty)}\Theta(t)\big)\,dt=0$, which is impossible since $\operatorname{Re}(1-\overline{\Theta(\infty)}\Theta)>0$ almost everywhere on $\mathbb{R}$ for any nonconstant inner function Θ. $\square$

6. The case of an arbitrary spectral multiplicity

Let U be a unitary part in the Wold–Kolmogorov decomposition (13) of the arbitrary cogenerator of the cocyclic perturbation. Any unitary operator U can be presented as an at most countable sum

$$U=\oplus_k U_k,$$

where the operators U_k are unitarily equivalent to the operators of multiplication in the appropriate spaces $L^2(\mu_k)$ and μ_k are measures on the circle $\mathbb{T}$,

$$(U_kf)(z)=zf(z),\quad f\in L^2(\mu_k).$$

Multiplying by positive weights, decreasing rapidly near the point 1, we can choose measures μ_k such that the condition

$$\sum_k\bigg(\int\limits_{\mathbb{T}}\frac{d\mu_k(\xi)}{|1-\xi|^q}\bigg)^{1/q}<\infty \tag{27}$$

holds for all $q>0$. Let us define the inner functions θ_k, connected with the measures μ_k by the formula (15). Condition (27) ensures that the product $\prod_k\theta_k$ converges to some inner function θ. Put

$$\hat\theta_n=\prod_{k=1}^{n-1}\theta_k$$

and define the cogenerator $\tilde V$ by the formula

$$\tilde V=\tilde S+\sum_n(\cdot,\hat\theta_ng_n)\hat\theta_n(1-\theta_n)-(\cdot,\overline{z})(1-\overline{\theta(1)}\theta),$$

where

$$g_n(z)=\frac{\theta_n(z)-\theta(0)}{z(1-\theta_n(0))}.$$

Proof of Theorem 1. The operator $V=\tilde V|_K$ is diagonal with respect to the orthogonal decomposition $K=\oplus_k\hat\theta_kK_{\theta_k}\oplus\theta K$. Condition (27) and Proposition 8 entail that

$$\varphi_t(V)-\varphi_t(S)\in\mathfrak{S}_1,\qquad t\ge 0.$$

The same condition (27) and Proposition 9 provide the inclusion

$$\varphi_t(\tilde V)-\varphi_t(\tilde S)\in\mathfrak{S}_p,\qquad t\ge 0,$$

for $p > q'$. Since the condition (27) holds for arbitrarily large values of q by the choice of measures, we have $\varphi_t(\tilde{V}) - \varphi_t(\tilde{S}) \in \mathfrak{S}_p$ for any $p > 1$. □

Proof of Theorem 2. Let U be a cogenerator of an arbitrary semigroup of unitary operators, being a unitary part in the Wold–Kolmogorov decomposition of the cocyclic perturbation. Then, there is an operator Δ, belonging to all classes $\mathfrak{S}_p$ for $p > 1$, that the perturbation $U + \Delta$ has a singular spectrum (see [11]). Moreover,

$$\varphi_t(U + \Delta) - \varphi_t(U) \in \mathfrak{S}_p,\ t \geq 0.$$

A detailed proof of the last statement is given in [5] (proof of Theorem 1.3). To complete the proof it is sufficient to apply Theorem 1. □

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